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A COMPACT TOPOLOGY FOR σ-ALGEBRA CONVERGENCE

PATRICK BEISSNER AND JONAS M. TÖLLE

Abstract. We propose a sequential topology on the collection of sub-σ-algebras included in a separable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We prove compactness of the conditional expectations with respect to L^2-bounded random variables along sequences of sub-σ-algebras. The varying index of measurability is captured by a bundle space construction. As a consequence, we establish the compactness of the space of sub-σ-algebras. The proposed topology preserves independence and is compatible with join and meet operations. Finally, a new application to information economics is discussed.

1. Introduction

Topological structures in probability theory enter at any stage. Results on compactness for the state space and probability measures can be found in most textbooks. In contrast to that, results on the compactness of the collection of sub-σ-algebras are to the best of our knowledge not available. This paper aims to fill this gap, by introducing a topology that employs the one-to-one correspondence between sub-σ-algebras and certain closed Hilbert subspaces of L^2. In particular, this allows us to borrow functional analytic convergence notions of sets, à la Mosco [30]. At the same time, the convergence is formulated in probabilistic terms via conditional expectations.

Departing from the standard measure theoretic setup, we first introduce the L^2-*varying convergence* of σ-algebras that is a convergence of norms of conditional expectations being tested by elements from L^2. The induced sequential topology is Hausdorff and metrizable. Our compactness result Theorem 4.1 relies on weak compactness (extending the Banach-Alaoglu theorem) in a fiber bundle structure that is now parametrized by the indexing set of all sub-σ-algebras. Within this abstract framework, the main step is to show that any limit point of any converging net in the bundle structure can be identified with an orthogonal projection operator.

To show compactness in the bundle structure, we consider the disjoint collection of unit balls within copies of L^2-spaces, which are again indexed by sub-σ-algebras. Using ideas of Kuwae and Shioya [27, 28], we define strong and weak convergence for sequences of functions "along" L^2-varying indexing sequences. In particular, we show that, in our case, both of the induced topologies are actually of sequential or even metric type. The L^2-subspaces converge Mosco, see Attouch [6] for this notion. In a further step, we embed this bundle space into an infinite product of compact spaces (via Tychonoff, as in the standard proof of Prohorov's theorem). Starting with an arbitrary sequence of sub-σ-algebras, we identify at least one σ-algebra as a limit for a convergent subsequence. Finally, the continuity of the projection from the weakly compact bundle of (uniformly L^2-bounded) random variables to the index space of sub-σ-algebras yields the desired compactness.

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Our single assumption on the probability space for receiving the compactness is separability. At the same time, the topology of L^2-varying convergence departs from the analytical and geometric L^2-structure of conditional expectations. One advantage is then the fruitful incorporation of functional analytic tools, such as the use of the mentioned method of Mosco type convergence in combination with a bundle space construction. To make this precise we recall a crucial result from Schilling [35, Theorem 25.2]. In a nutshell, we may arbitrarily switch between the members of the following classes related to a given L^2-space under a probability space:

- sub-σ-algebras;
- a certain class of closed linear subspaces[1] of L^2;
- the conditional expectation operator[2];
- sub-Markovian orthogonal projection operators[3].

From this perspective, it seems that the compactness of the set of sub-σ-algebras is an intrinsic property under the structure of the L^2-space. Another justification is established, by showing the comparability of the join/meet operation and the stability of independence in the limit.

The closest prior result to ours can be found in Artstein [5], where the notion of a conditional expectation is relaxed in the Young measure sense and a compact convergence is observed. In contrast to that, our setup guarantees that the each limit point can be identified with a sub-σ-algebra. Recently, Tsirelson [39] relies on similar grounding for questions of classicality.

The present L^2-varying convergence of norms for conditional expectations, tested by the elements of a set of random variables, can be nested into other notions of convergence. Variations rely, on the one hand, on the change of the space of tested random variables and, on the other hand, on the type of convergence of random variables. A hierarchy of implications between the various types of convergence is inherited, see Neveu [32], Kudō [26], Alonso and Brambila-Paz [3] for such variations. Stronger topologies, among others, employ Hausdorff convergence (see Boylan [12], Rogge [34], Landers and Rogge [29], Van Zandt [41]) or a set-theoretic notion (see Fetter [19]).

In economics, the concept of a σ-algebra serves as a model for information. This has initiated a program in suggesting meaningful topologies on the set of sub-σ-algebras, Khan et al. [24], Stinchcombe [36], Cotter [15], Allen [2]. For an application in martingale theory, see Coquet et al. [14]. However, our compactness result opens the door for new applications in the economics of information. We illustrate this by a model of information design. Communication in a game theoretic setting can now be modeled by an information designer (see Bergemann and Morris [8]), who chooses the information transfer by means of the resulting value of information.

The paper is organized as follows. Section 2 presents the L^2-varying convergence. Section 3 introduces and discusses the topological setting. Section 4 introduces and proves the main result. Section 5 shows that our topology is compatible with join/meet operations and independence. Section 6 presents an application to information economics. The Appendix recalls some facts from general topology and contains postponed proofs.

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[1]That is, precisely the class of closed subspaces spanned by the closure of an algebra of bounded functions.

[2]Which is precisely the orthogonal projection operator which maps onto a closed linear subspace of L^2 with the property that it is itself an L^2-space w.r.t. a sub-σ-algebra.

[3]Note that conditional expectations are characterized as projections with range being a lattice, see Andô [4], Bernau and Lacey [9].

2. Convergence of sub-σ-algebras

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a separable probability space. Following the terminology in Bogachev [11, 7.14 (iv)], we call $(\Omega, \mathcal{F}, \mathbb{P})$ *separable* if $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is separable in the Hilbert space sense.

Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ be σ-algebras. Set $\mathcal{A} \vee \mathcal{B} := \sigma(\mathcal{A} \cup \mathcal{B})$ (the intersection of all σ-algebras containing $\mathcal{A}$ and $\mathcal{B}$, called *join* of $\mathcal{A}$ and $\mathcal{B}$) and $\mathcal{A} \wedge \mathcal{B} := \mathcal{A} \cap \mathcal{B}$ (called *meet* of $\mathcal{A}$ and $\mathcal{B}$). Denote by $\mathcal{N} := \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$ the σ-ideal of $\mathbb{P}$-null sets from $\mathcal{F}$. Let $\mathbb{F} := \{\mathcal{A} \subseteq \mathcal{F} : \mathcal{A} \text{ is a } \sigma\text{-algebra}\}$. Let $\mathbb{F}^* := \{\mathcal{A} \vee \mathcal{N} : \mathcal{A} \in \mathbb{F}\}$, which we call the *collection of all sub-σ-algebras* of $(\Omega, \mathcal{F}, \mathbb{P})$. Clearly, for every $\mathcal{A} \in \mathbb{F}^*$, $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and $L^2(\mathcal{A}) := L^2(\Omega, \mathcal{A}, \mathbb{P})$ is a closed Hilbert subspace of $L^2(\mathcal{F})$ with orthogonal projection given by the conditional expectation $f \mapsto \mathbb{E}[f|\mathcal{A}] =: P_{\mathcal{A}}(f)$. Set $\|\cdot\|_p := \|\cdot\|_{L^p(\mathcal{F})}$, $p \in [1, \infty]$ and let $\langle f, g\rangle := \mathbb{E}[fg]$, whenever the right-hand side is well-defined and finite for some $f, g \in L^1(\mathcal{F})$.

We focus on the following notion for σ-algebra convergence.

Definition 2.1. Let $\mathcal{B}_n \in \mathbb{F}^*$, $n \in \mathbb{N}$, $\mathcal{B} \in \mathbb{F}^*$. We say that $\mathcal{B}_n \to \mathcal{B}$ *in the L^2-varying sense* as $n \to \infty$, if

$$\|\mathbb{E}[f|\mathcal{B}_n]\|_2 \to \|\mathbb{E}[f|\mathcal{B}]\|_2$$

as $n \to \infty$ for every $f \in L^2(\mathcal{F})$.

We illustrate the convergence in the following example, and furthermore, prove that the modification toward almost sure convergence results in a counterexample.

Example 2.2. Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, that is, the Borel σ-algebra on $[0,1]$ and $\mathbb{P} = dx \llcorner [0,1]$ is the Lebesgue measure restricted to $[0,1]$. Define the sequence $\mathcal{B}_n = \sigma(I^{(n)}) \vee \mathcal{N}$, where $I^{(n)} = I(n - 2^{\lfloor \log_2(n) \rfloor}, \lfloor \log_2(n) \rfloor)$ and $I(k,m) = \left[\frac{k}{2^m}, \frac{k+1}{2^m}\right]$ for $0 \le k \le 2^m - 1, m \in \mathbb{N}$.

Claim 1. $\mathcal{B}_n \to \{\varnothing, \Omega\} \vee \mathcal{N} =: \mathcal{B}_0$ converges L^2-varying as $n \to \infty$.

Proof of Claim 1. To see this, let $f \in L^2(\mathcal{F})$. We have that,

$$\|\mathbb{E}[f|\mathcal{B}_n]\|_2^2 = \mathbb{E}\left[\frac{\mathbb{E}[f 1_{I^{(n)}}]^2 1_{I^{(n)}}}{\mathbb{P}(I^{(n)})^2}\right] + \mathbb{E}\left[\frac{\mathbb{E}[f 1_{(I^{(n)})^c}]^2 1_{(I^{(n)})^c}}{(1 - \mathbb{P}(I^{(n)}))^2}\right].$$

Note that for $p \in [1, \infty)$, $\mathbb{E}[1^p_{I^{(n)}}] = \mathbb{P}(I^{(n)}) = 2^{-\lfloor \log_2 n \rfloor} \le 2^{-\log_2(n)+1} = \frac{2}{n} \to 0$ as $n \to \infty$. For the same reason, $1_{I^{(n)}} \to 0$ strongly in $L^p(\mathcal{F})$ for any $p \in [1, \infty)$. For a subsequence $\{n_k\}$, let $g \in L^\infty(\mathcal{F})$ be a weak* limit of the sequence $\{1_{I^{(n_k)}}\}$. However, $\langle g, h\rangle = \lim_k \langle 1_{I^{(n_k)}}, h\rangle = 0$ for any $h \in L^q(\mathcal{F})$, $q > 1$ and hence $g = 0$. As this argument works for any further extracted subsequence, we conclude that $1_{I^{(n)}} \to 0$ weak* in $L^\infty(\mathcal{F})$. Also, it follows that $1_{(I^{(n)})^c} \to 1_\Omega$ strongly in $L^2(\mathcal{F})$.

Via $\mathbb{E}[f|\mathcal{B}_0] = \mathbb{E}[f]1_\Omega$ and $\|\mathbb{E}[f|\mathcal{B}_0]\|_2^2 = \mathbb{E}[f]^2$, we apply Jensen's inequality for the measure $\tilde{P}_n(\cdot) = \mathbb{P}(\,\cdot\,|I^{(n)})$,

$$\begin{aligned}
& \left|\|\mathbb{E}[f|\mathcal{B}_n]\|_2^2 - \|\mathbb{E}[f|\mathcal{B}_0]\|_2^2\right| \\
= \quad & \left|\mathbb{E}\left[\frac{\mathbb{E}[f1_{I^{(n)}}]^2 1_{I^{(n)}}}{\mathbb{P}(I^{(n)})^2}\right] + \frac{\mathbb{E}[f1_{(I^{(n)})^c}]^2}{1-\mathbb{P}(I^{(n)})} - \mathbb{E}[f]^2\right| \\
\leq \quad & \mathbb{P}(I^{(n)}) \int_{I^{(n)}} f^2\, d\tilde{P}_n + \left|\frac{\mathbb{E}[f1_{(I^{(n)})^c}]^2}{1-\mathbb{P}(I^{(n)})} - \mathbb{E}[f]^2\right| \\
= \quad & \underbrace{\langle f^2, 1_{I^{(n)}}\rangle}_{\xrightarrow[n\to\infty]{} 0} + \left|\frac{\langle f, 1_{(I^{(n)})^c}\rangle^2}{1-\mathbb{P}(I^{(n)})} - \mathbb{E}[f]^2\right| \\
\xrightarrow[n\to\infty]{} \quad & 0 + \left|\langle f, 1_\Omega\rangle^2 - \mathbb{E}[f]^2\right| = 0.
\end{aligned}$$

□

Claim 2. There exists $g_0 \in L^2(\mathcal{F})$ such that $\mathbb{E}[g_0|\mathcal{B}_n] \not\to \mathbb{E}[g_0|\mathcal{B}_0]$ $\mathbb{P}$-a.s. We note, however, that for the same g_0, there exists a (fast) subsequence $\{\mathcal{B}_{n_k}\}$ such that $\mathbb{E}[g_0|\mathcal{B}_{n_k}] \to \mathbb{E}[g_0|\mathcal{B}_0]$ $\mathbb{P}$-a.s.

Proof of Claim 2. One possible choice is $g_0(\omega) := 2\omega$, $\omega \in [0,1]$. We postpone the lengthy but straightforward proof to Appendix B.

□

Lemma 2.3. *Let $\mathcal{B}_n, \mathcal{B} \in \mathbb{F}^*$, $n \in \mathbb{N}$. Suppose that $\mathcal{B}_n \to \mathcal{B}$ in the L^2-varying sense. Then*

$$\|\mathbb{E}[f|\mathcal{B}_n] - \mathbb{E}[f|\mathcal{B}]\|_2 \to 0$$

as $n \to \infty$ for every $f \in L^2(\mathcal{F})$.

Proof. Fix $f \in L^2(\mathcal{F})$. In Hilbert spaces, convergence of norms together with weak convergence implies strong convergence. Therefore, it is sufficient to prove that the random variables $P_n f := \mathbb{E}[f|\mathcal{B}_n]$, $n \in \mathbb{N}$, $Pf := \mathbb{E}[f|\mathcal{B}]$ converge weakly $P_n f \rightharpoonup Pf$ in the Hilbert space sense in $L^2(\mathcal{F})$ as $n \to \infty$. The conditional expectation $P_\mathcal{A} : f \mapsto \mathbb{E}[f|\mathcal{A}]$ is a linear orthogonal projection operator onto $L^2(\mathcal{A})$ in $H := L^2(\mathcal{F})$ for any $\mathcal{A} \in \mathbb{F}^*$. By the sequential Banach-Alaoglu theorem, the unit ball in the space of bounded linear operators[4] $L(L^2(\mathcal{F}))$ is sequentially compact with respect to the weak operator topology. However, since the range of P_n, $n \in \mathbb{N}$ is always at least one-dimensional, $\|P_n\|_{L(L^2(\mathcal{F}))} = 1$ and thus a subsequence $\{P_{n_k}\}$ converges in the weak operator topology to some bounded linear operator Q. Let $g \in L^2(\mathcal{F})$. We get by the polarization identity that

$$\langle Qf, g\rangle = \lim_k \langle P_{n_k} f, g\rangle = \lim_k \langle P_{n_k} f, P_{n_k} g\rangle = \langle Pf, Pg\rangle = \langle Pf, g\rangle.$$

Hence, $Q = P$ and, since the argument works for any further subsequence, the initial sequence converges $P_n f \rightharpoonup Pf$ weakly for all $f \in L^2(\mathcal{F})$. □

Notation. Due to Proposition 2.4 below, L^2-varying convergence induces a topology on $\mathbb{F}^*$ which we denote by κ.

[4]For a normed space X, we set $L(X) := L(X,X)$ and denote by it the space of all linear and bounded operators $T : X \to X$ with operator norm $\|T\|_{L(X)} := \sup_{\|x\|_X \leq 1} \|Tx\|_X$.

Proposition 2.4. *The following metric generates the topology of L^2-varying convergence*

$$d_\kappa(\mathcal{A},\mathcal{B}) := \sum_{j=1}^{\infty} 2^{-j} \frac{\Big| \|\mathbb{E}[f_j|\mathcal{A}]\|_2 - \|\mathbb{E}[f_j|\mathcal{B}]\|_2 \Big|}{1 + \Big| \|\mathbb{E}[f_j|\mathcal{A}]\|_2 - \|\mathbb{E}[f_j|\mathcal{B}]\|_2 \Big|}, \quad \mathcal{A},\mathcal{B} \in \mathbb{F}^*,$$

where $\{f_j\}_{j\in\mathbb{N}}$ is a countable dense subset of $L^2(\mathcal{F})$.

Proof. It is a standard exercise to prove that d_κ defines a pseudometric on $\mathbb{F}^*$. It indeed defines a metric, as $d_\kappa(\mathcal{A},\mathcal{B}) = 0$ implies

$$\|\mathbb{E}[f_j|\mathcal{A}]\|_2 = \|\mathbb{E}[f_j|\mathcal{B}]\|_2 \tag{2.1}$$

for every $j \in \mathbb{N}$. The conditional expectation $P_\mathcal{A} : f \mapsto \mathbb{E}[f|\mathcal{A}]$ is a linear orthogonal projection operator on $L^2(\mathcal{F})$ for any $\mathcal{A} \in \mathbb{F}^*$. Thus (2.1) implies $\|P_\mathcal{B} f_j\|_2 = \|P_{\tilde{\mathcal{B}}} f_j\|_2$ for all $j \in \mathbb{N}$. It follows by e.g. Kubrusly [25, Problem 2.9] and a density argument that $P_\mathcal{B} = P_{\tilde{\mathcal{B}}}$ and hence that $L^2(\mathcal{B}) = L^2(\tilde{\mathcal{B}})$ as closed subspaces of $L^2(\mathcal{F})$, which yields that $\mathcal{B} = \tilde{\mathcal{B}}$ up to $\mathbb{P}$-negligible elements of $\mathcal{F}$.

It remains to show that d_κ generates the L^2-varying topology κ. Clearly, $\lim_n d_\kappa(\mathcal{B}_n,\mathcal{B}) = 0$ if and only if $\lim_n \|\mathbb{E}[f_j|\mathcal{B}_n]\|_2 = \|\mathbb{E}[f_j|\mathcal{B}]\|_2$ for each $j \in \mathbb{N}$. We claim that this equivalent to L^2-varying convergence $\mathcal{B}_n \to \mathcal{B}$. In order to see this, let $f \in L^2(\mathcal{F})$ and let $f^k \in \{f_j\}_{j\in\mathbb{N}}$, $k \in \mathbb{N}$ such that $\lim_k \|f^k - f\|_2 = 0$. We get that

$$\begin{aligned}
&|\|\mathbb{E}[f|\mathcal{B}_n]\|_2 - \|\mathbb{E}[f|\mathcal{B}]\|_2| \\
\le& \big|\|\mathbb{E}[f|\mathcal{B}_n]\|_2 - \|\mathbb{E}[f^k|\mathcal{B}_n]\|_2\big| + \big|\|\mathbb{E}[f^k|\mathcal{B}]\|_2 - \|\mathbb{E}[f|\mathcal{B}]\|_2\big| \\
&+ \big|\|\mathbb{E}[f^k|\mathcal{B}_n]\|_2 - \|\mathbb{E}[f^k|\mathcal{B}]\|_2\big| \\
\le& \|\mathbb{E}[|f - f^k||\mathcal{B}_n]\|_2 + \|\mathbb{E}[|f - f^k||\mathcal{B}]\|_2 + \big|\|\mathbb{E}[f^k|\mathcal{B}_n]\|_2 - \|\mathbb{E}[f^k|\mathcal{B}]\|_2\big| \\
\le& 2\|f - f^k\|_2 + \big|\|\mathbb{E}[f^k|\mathcal{B}_n]\|_2 - \|\mathbb{E}[f^k|\mathcal{B}]\|_2\big|.
\end{aligned}$$

Thus, letting first $n \to \infty$ and then $k \to \infty$ yields the claim. □

As a metric space, $(\mathbb{F}^*, \kappa)$ is a first-countable Hausdorff space.

3. Convergence in bundle spaces

Denote by[5]

$$\mathbb{H} := \bigsqcup_{\mathcal{B}\in\mathbb{F}^*} L^2(\mathcal{B})$$

the disjoint union of L^2-spaces, indexed by the sub-σ-algebras of $\mathcal{F}$. Let

$$\pi : \mathbb{H} \to \mathbb{F}^*$$

[5]Let I be an index set and let A_i, $i \in I$ be sets. Then the *disjoint union* is defined as the following set of pairs

$$\bigsqcup_{i\in I} A_i := \bigcup_{i\in I} \{(x,i) \mid x \in A_i\}.$$

By abuse of notation, for $x \in \bigsqcup_{i\in I} A_i$, we drop the reference to its index i in the notation, assuming quietly that we are actually considering a pair (x,i) and not just an element x. We may recover the index from any $x \in \bigsqcup_{i\in I} A_i$ by introducing the map $\pi(x) := i$, whenever $x \in A_i$, that is, the projection on the index element.

be the (bundle) projection on the index of the element in $\mathbb{H}$. The collection $\mathbb{H}$ mimics the total space of a fiber bundle, whereas $\mathbb{F}^*$ plays the role of the base space of a fiber bundle:

$$\begin{tikzcd} \mathbb{H} \arrow[r, "\iota"] \arrow[d, "\pi"'] & L^2(\mathcal{F}) \times \mathbb{F}^* \arrow[dl, "\mathrm{proj}_2"] \\ \mathbb{F}^* & \end{tikzcd} \tag{3.1}$$

Here,

$$\iota : \mathbb{H} \to L^2(\mathcal{F}) \times \mathbb{F}^*, \quad \iota(u) := (u, \pi(u)) \tag{3.2}$$

denotes the standard embedding and proj_2 denotes the projection on the second component.

We usually require to take copies of elements in L^2, that is, for instance the constant function $1_\Omega \in L^2(\{\varnothing, \Omega\})$ is clearly distinguished from the constant function $1_\Omega \in L^2(\mathcal{F})$, whenever $\mathcal{F}$ is non-trivial. This is accomplished by implicitly keeping track of the index element $\pi(1_\Omega) \in \mathbb{F}^*$ (which is obviously not necessarily the same object as $\sigma(1_\Omega)$).

Note, however, that we are not in the case of fiber bundles as e.g. in Husemöller [21], as the fiber spaces $L^2(\mathcal{B})$ might be either finite or infinite dimensional so that a universal isomorphic fiber space (a candidate would be $L^2(\mathcal{F})$) does not necessarily exist. Note also that we do neither assume nor verify that $(\mathbb{F}^*, \kappa)$ is connected. On these lines, see also Dupré [16].

Next we define strong (denoted by τ) and weak (denoted by σ) topologies on $\mathbb{H}$ that coincide with the original strong and weak topologies on each "fiber" $L^2(\mathcal{B})$, $\mathcal{B} \in \mathbb{F}^*$ and that capture strong and weak convergence "along" a sequence L^2-spaces associated to a L^2-varying convergent sequence of sub-σ-algebras, compare e.g. with Kuwae and Shioya [27, 28], Tölle [38]. Both topologies rely on convergence of sequences, see also Appendix A.

3.1. **Strong convergence.** Without the bundle structure, the following type of convergence (for nets replacing sequences) in general Hilbert spaces can be found in Kuwae and Shioya [27]. On an earlier comparable approach, see Stummel [37].

Definition 3.1. Let $u_k \in \mathbb{H}$, $k \in \mathbb{N}$, $u \in \mathbb{H}$. We say that $u_k \to u$ *strongly* if $\pi(u_k) \to \pi(u)$ L^2-varying and there exist elements $\tilde{u}_m \in L^2(\pi(u))$, $m \in \mathbb{N}$, such that $\|\tilde{u}_m - u\|_2 \to 0$ as $m \to \infty$ and

$$\lim_m \limsup_k \|u_k - \mathbb{E}[\tilde{u}_m | \pi(u_k)]\|_2 = 0. \tag{S}$$

Remark 3.2. Assume that we are given $u_k \in L^2(\mathcal{B})$, $k \in \mathbb{N}$, $u \in L^2(\mathcal{B})$ for some fixed $\mathcal{B} \in \mathbb{F}^*$. Then $u_k \to u$ strongly in $L^2(\mathcal{B})$ if and only if $u_k \to u$ strongly in the sense of Definition 3.1.

Lemma 3.3. *Let $u \in L^2(\mathcal{F})$ and let $\mathcal{B}_n \to \mathcal{B}$ in the L^2-varying sense. Then $\mathbb{E}[u|\mathcal{B}_n] \to \mathbb{E}[u|\mathcal{B}]$ strongly.*

Proof. Set $\tilde{u}_m := \mathbb{E}[u|\mathcal{B}]$ for every $m \in \mathbb{N}$. Then by L^2-varying convergence of $\mathcal{B}_n \to \mathcal{B}$, we get that

$$\|\mathbb{E}[u|\mathcal{B}_n] - \mathbb{E}[\mathbb{E}[u|\mathcal{B}]|\mathcal{B}_n]\|_2 = \|\mathbb{E}[u - \mathbb{E}[u|\mathcal{B}]|\mathcal{B}_n]\|_2 \to \|\mathbb{E}[u - \mathbb{E}[u|\mathcal{B}]|\mathcal{B}]\|_2 = 0.$$

□

We get the following:

Corollary 3.4 (Existence of strongly convergent sequences)**.** *Let $\mathcal{B}_k, \mathcal{B} \in \mathbb{F}^*$, $k \in \mathbb{N}$. Suppose that $\mathcal{B}_k \to \mathcal{B}$ in the L^2-varying sense. Then for each $u \in L^2(\mathcal{B})$ there exist $u_k \in L^2(\mathcal{B}_k)$, $k \in \mathbb{N}$ such that $u_k \to u$ strongly in $\mathbb{H}$.*

Lemma 3.5. *Let $u_k, u \in \mathbb{H}$, $k \in \mathbb{N}$. Suppose that $\pi(u_k) \to \pi(u)$ in the L^2-varying sense. Then the following conditions are equivalent*

(i) $u_k \to u$ strongly in $\mathbb{H}$,
(ii) $\lim_n \|u_k - \mathbb{E}[u|\pi(u_k)]\|_2 = 0$,
(iii) $\lim_k \|u_k - u\|_2 = 0$.

In particular, each strongly convergent sequence possesses exactly one limit.

Proof. Suppose that (i) holds. Let $\tilde{u}_m \in L^2(\pi(u))$, $m \in \mathbb{N}$ be such that $\|\tilde{u}_m - u\|_2 \to 0$. Then

$$\begin{aligned}\|u_k - \mathbb{E}[u|\pi(u_k)]\|_2 \leq& \|u_k - \mathbb{E}[\tilde{u}_m|\pi(u_k)]\|_2 + \|\mathbb{E}[\tilde{u}_m - u|\pi(u_k)]\|_2 \\ \leq& \|u_k - \mathbb{E}[\tilde{u}_m|\pi(u_k)]\|_2 + \|\tilde{u}_m - u\|_2,\end{aligned}$$

which tends to zero by (i). Thus (i) $\Longrightarrow$(ii).

Suppose that (ii) holds. We get that

$$\|u_k - u\|_2 \leq \|u_k - \mathbb{E}[u|\pi(u_k)]\|_2 + \|\mathbb{E}[u|\pi(u_k)] - u\|_2,$$

where the first part tends to zero by (ii) and the second part tends to zero by Lemma 2.3. Thus (ii) $\Longrightarrow$(iii).

Suppose that (iii) holds. Clearly, by setting $\tilde{u}_m := u$ for every $m \in \mathbb{N}$, we get that

$$\|u_k - \mathbb{E}[u|\pi(u_k)]\|_2 \leq \|u_k - u\|_2 + \|u - \mathbb{E}[u|\pi(u_k)]\|_2,$$

where the first part tends to zero by (iii) and the second part tends to zero by Lemma 2.3. Thus (iii) $\Longrightarrow$(i).

The proof is complete. □

Notation. Due to Proposition 3.6 below, strong convergence induces a topology on $\mathbb{H}$ which we denote by τ.

Proposition 3.6. *Strong convergence is an $\mathcal{S}^*$-sequential convergence in the sense of Definition A.4, and thus generates a Fréchet-Urysohn topology on $\mathbb{H}$.*

Proof. See Appendix A.2. □

Remark 3.7. In addition, strong convergence is metrizable with metric

$$d_\tau(u,v) := \|u - v\|_2 + d_\kappa(\pi(u), \pi(v)) \quad u, v \in \mathbb{H},$$

compare with Proposition 2.4. Thus, $\mathbb{H}$ can be identified with a closed metric subspace $\iota(\mathbb{H}) \subseteq L^2(\mathcal{F}) \times \mathbb{F}^*$, see (3.2) above.

Lemma 3.8 (Properties of strong convergence)**.** *Let $\alpha, \beta \in \mathbb{R}$, $u_k, v_k, u, v \in \mathbb{H}$, $k \in \mathbb{N}$.*

(i) *If $u_k \to u$ strongly, then $\|u_k\|_2 \to \|u\|_2$ as $k \to \infty$.*
(ii) *If $u_k \to u$ strongly and $\pi(v_k) = \pi(u_k)$ for all $k \in \mathbb{N}$, then $v_k \to u$ if and only if $\|u_k - v_k\|_2 \to 0$ as $k \to \infty$.*
(iii) *If $u_k \to u$ strongly, $\pi(v_k) = \pi(u_k)$ for all $k \in \mathbb{N}$, and $v_k \to v$ strongly, then $\alpha u_k + \beta v_k \to \alpha u + \beta v$ strongly.*

Proof. Let $\alpha, \beta \in \mathbb{R}$, $u_k, v_k, u, v \in \mathbb{H}$, $k \in \mathbb{N}$. Assume that $u_k \to u$ strongly.

(i): Follows by
$$|\|u_k\|_2 - \|u\|_2| \leq \|u_k - u\|_2$$
and Lemma 3.5 (iii).

(ii): By the assumption, $\pi(v_k) = \pi(u_k) \to \pi(u)$ in the L^2-varying sense. Hence, the claim follows by an $\varepsilon/2$-argument and Lemma 3.5 (iii).

(iii): Taking the proof of (ii) and the linearity of the conditional expectation into account, the claim follows by standard arguments.

□

3.2. **Weak convergence.** In the following, we introduce the weak convergence on $\mathbb{H}$.

Definition 3.9. Let $u_k, u \in \mathbb{H}$, $k \in \mathbb{N}$. We say that $u_k \rightharpoonup u$ *weakly* if $\pi(u_k) \to \pi(u)$ in the L^2-varying sense and the following two conditions are satisfied:

(W1) it holds that
$$\sup_k \|u_k\|_2 < +\infty,$$
(W2) and, we have that
$$\lim_k \langle u_k, v_k \rangle = \langle u, v \rangle$$
for all $v_k \in L^2(\pi(u_k))$, $k \in \mathbb{N}$, $v \in L^2(\pi(u))$ such that $v_k \to v$ strongly in $\mathbb{H}$.

Remark 3.10. Assume that we are given $v_k \in L^2(\mathcal{B})$, $k \in \mathbb{N}$, $v \in L^2(\mathcal{B})$ for some fixed $\mathcal{B} \in \mathbb{F}^*$. Then $v_k \rightharpoonup v$ weakly in $L^2(\mathcal{B})$ if and only if $v_k \rightharpoonup v$ weakly in the sense of Definition 3.9.

Remark 3.11. A weakly convergent sequence $\{u_k\}$ in $\mathbb{H}$ possesses exactly one limit, which is seen by Definition 3.9 (W2), namely, for some limit u and any other possible limit $\tilde{u}$, we get by L^2-varying convergence that $\pi(u) = \pi(\tilde{u})$ and that $\langle u - \tilde{u}, v \rangle = 0$ for every $v \in L^2(\pi(u))$.

Notation. Due to Proposition 3.12 below, weak convergence indeed induces a topology on $\mathbb{H}$ which we denote by σ.

Proposition 3.12. *Weak convergence is an $\mathcal{L}^*$-sequential convergence in the sense of Definition A.4, and thus generates a sequential topology on $\mathbb{H}$.*

Proof. See Appendix A.2. □

Lemma 3.13 (Properties of weak convergence)**.** *Let $\alpha, \beta \in \mathbb{R}$, $u_k, v_k, u, v \in \mathbb{H}$, $k \in \mathbb{N}$.*

(i) *If $u_k \to u$ strongly, then $u_k \rightharpoonup u$ weakly as $k \to \infty$.*
(ii) *If $u_k \rightharpoonup u$ weakly, then $\liminf_k \|u_k\|_2 \geq \|u\|_2$.*
(iii) *If $u_k \rightharpoonup u$ weakly, $\pi(v_k) = \pi(u_k)$ for all $k \in \mathbb{N}$, and $v_k \rightharpoonup v$ weakly, then $\alpha u_k + \beta v_k \rightharpoonup \alpha u + \beta v$ weakly.*

Proof. Let $\alpha, \beta \in \mathbb{R}$, $u_k, v_k, u, v \in \mathbb{H}$, $k \in \mathbb{N}$.

(i): Assume that $u_k \to u$ strongly. L^2-varying convergence is immediate. (W1) follows from Lemma 3.8 (i), as does (W2) by employing the polarization identity.

(ii): Assume that $u_k \rightharpoonup u$ weakly. By (W1), we get that $\liminf_k \|u_k\|_2 < \infty$. It follows that there is a subsequence $\{u_{k_l}\}$ of $\{u_k\}$ such that $\lim_l \|u_{k_l}\|_2 = \liminf_k \|u_k\|_2$. Clearly, by Proposition 3.12, $u_{k_l} \rightharpoonup u$ weakly, too. We can find $u_0 \in L^2(\pi(u))$ with $\|u_0\|_2 = 1$ and $\langle u_0, u \rangle = \|u\|_2$. By Lemma 3.3, $\mathbb{E}[u_0|\pi(u_k)] \to u_0$ strongly as $k \to \infty$. By Lemma 3.8 (i) and (W2), we get that
$$\begin{aligned}\liminf_k \|u_k\|_2 &= \underbrace{\lim_l \|\mathbb{E}[u_0|\pi(u_k)]\|_2}_{=1} \lim_k \|u_{k_l}\|_2 \\ &\geq \lim_k \langle \mathbb{E}[u_0|\pi(u_k)], u_{k_l} \rangle = \langle u_0, u \rangle = \|u\|_2.\end{aligned}$$

(iii): The part for the L^2-varying convergence is clear. (W1) follows by the triangle inequality. (W2) follows by the bilinearity of the scalar product.

□

Lemma 3.14. *Let $u_k, u \in \mathbb{H}$, $k \in \mathbb{N}$. Then the following statements are equivalent.*

(i) *$u_k \to u$ strongly.*
(ii) *$u_k \rightharpoonup u$ weakly and $\|u_k\|_2 \to \|u\|_2$ as $k \to \infty$.*

Proof. Assume (i). Then (ii) follows from Lemma 3.13 (i) and Lemma 3.8 (i) respectively. L^2-varying convergence follows, too.

Assume (ii). Clearly, by Lemma 3.3,

$$\lim_k \langle u_k, \mathbb{E}[u|\pi(u_k)]\rangle = \langle u, u\rangle = \|u\|_2^2.$$

Furthermore, by Lemma 3.8 (i),

$$\begin{aligned} 0 \leq & \lim_k \|u_k - \mathbb{E}[u|\pi(u_k)]\|_2^2 \\ = & \lim_k \left(\|u_k\|_2^2 + 2\langle u_k, \mathbb{E}[u|\pi(u_k)]\rangle + \|\mathbb{E}[u|\pi(u_k)]\|_2^2\right) \\ = & \lim_k \|u_k\|_2^2 - \|u\|_2^2 = 0, \end{aligned}$$

and hence (i) follows from Lemma 3.5.

The proof is complete. □

4. Main result: compactness

We arrive at our main result.

Theorem 4.1. $(\mathbb{F}^*, \kappa)$ *is compact.*

We shall prove Theorem 4.1 in several steps. First of all, we shall introduce the bundle space of closed "unit balls",

$$\mathbb{H}_1 := \bigsqcup_{\mathcal{B}\in\mathbb{F}^*} \{f \in L^2(\mathcal{B}) \ : \ \|f\|_2 \leq 1\} \subset \mathbb{H}$$

that is, with fibers consisting of elements with norm less or equal to one. We shall denote the restriction of π to $\mathbb{H}_1$ by the same symbol. Define

$$\mathbb{T} := \prod_{u\in L^2(\mathcal{F})} ([-\|u\|_2, \|u\|_2] \times [0, \|u\|_2])$$

and equip $\mathbb{T}$ with the product topology, which is Hausdorff by Engelking [17, Theorem 2.3.11]. By Tychonoff's theorem, see Kelley [23, p. 143, Theorem 13], $\mathbb{T}$ is compact. A net $\{x_i\}_{i\in I}$ of elements in $\mathbb{T}$ converges to some $x \in \mathbb{T}$ if and only if

$$\lim_{i\in I} x_i(u) = x(u)$$

converges in $\mathbb{R}^2$ for any $u \in L^2(\mathcal{F})$.

The strategy to prove Theorem 4.1, inspired by Tölle [38, Theorem 5.22], is summarized in the following procedure:

- We define a map $\mathcal{I} : \mathbb{H}_1 \to \mathbb{T}$ (see (4.1) below) which verifies
  - $\mathcal{I}$ is injective, see Lemma 4.4 below.
  - $\mathcal{I}$ is a homeomorphism between $(\mathbb{H}_1, \sigma)$ and the range $\mathcal{K} := \mathcal{I}(\mathbb{H}_1) \subset \mathbb{T}$, carrying the relative topology, see Lemma 4.5 and Lemma 4.6 below.
- We prove that $\mathcal{K}$ is a closed subset of $\mathbb{T}$ and hence compact, see Proposition 4.7 below.
- We infer that $\mathbb{H}_1$ with the weak topology σ is a continuous image of a compact set and hence compact.

Remark 4.2. By definition, $\pi : \mathbb{H}_1 \to \mathbb{F}^*$ is both τ/κ-continuous as well as σ/κ-continuous, in other words, if $f_k, f \in \mathbb{H}_1$, $k \in \mathbb{N}$ with $f_k \to f$ strongly or $f_k \rightharpoonup f$ weakly, then it follows that $\pi(f_k) \to \pi(f)$ in L^2-varying sense.

Proof of Theorem 4.1. Taking into account the previous remark and the above procedure, we conclude that $\mathbb{F}^* = \pi(\mathbb{H}_1)$ is a continuous image of the compact space $(\mathbb{H}_1, \sigma)$, and thus is itself compact. Theorem 4.1 is proved. □

Corollary 4.3. *Let $\mathcal{B}_n \in \mathbb{F}^*$, $n \in \mathbb{N}$. Then there exists $\mathcal{B}_0 \in \mathbb{F}^*$ such that some subsequence $\{\mathcal{B}_{n_k}\}$ converges to $\mathcal{B}_0$ in L^2-varying sense.*

Proof. This follows from Theorem 4.1 by noting that in metric spaces (see Proposition 2.4), compactness implies sequential compactness. $\square$

4.1. **Remaining proof of Theorem 4.1.** Define $\mathcal{I} : \mathbb{H}_1 \to \mathbb{T}$ by

$$f \mapsto \mathcal{I}(f)(u) := \left(\langle f, \mathbb{E}[u|\pi(f)]\rangle, \|\mathbb{E}[u|\pi(f)]\|_2\right), \quad u \in L^2(\mathcal{F}). \tag{4.1}$$

The map $\mathcal{I}$ is well-defined. Let us denote by $(\cdot)_j : \mathbb{R}^2 \to \mathbb{R}$, $j \in \{1, 2\}$, the projection on the j-th component.

Lemma 4.4. *$\mathcal{I}$ is an injective map.*

Proof. Let $f, g \in \mathbb{H}_1$, $f \neq g$. Suppose first that $\pi(f) = \pi(g)$. Then there exists $u_0 \in L^2(\pi(f))$, $u_0 \neq 0$ with $\langle u_0, f\rangle \neq \langle u_0, g\rangle$. Clearly, $\langle u_0, f\rangle = \langle u_0, \mathbb{E}[f|\pi(f)]\rangle = (\mathcal{I}(f)(u_0))_1$. Similarly, $\langle u_0, g\rangle = \langle u_0, \mathbb{E}[g|\pi(g)]\rangle = (\mathcal{I}(g)(u_0))_1$. Suppose now, that $\pi(f) \neq \pi(g)$. Then there exists $v_0 \in L^2(\mathcal{F})$, $v_0 \neq 0$, such that $(\mathcal{I}(f)(v_0))_2 = \|\mathbb{E}[v_0|\pi(f)]\|_2 \neq \|\mathbb{E}[v_0|\pi(g)]\|_2 = (\mathcal{I}(g)(v_0))_2$. The injectivity follows. $\square$

Lemma 4.5. *The map $\mathcal{I}$ is continuous with respect to weak convergence in $\mathbb{H}_1$.*

Proof. By Lemma A.3 (iv) in the appendix, it is sufficient to prove continuity with the help of sequences. Let $f_l \in \mathbb{H}_1$, $l \in \mathbb{N}$ be a weakly convergent sequence with weak limit f, being unique by Remark 3.11. As a consequence, $\pi(f_l) \to \pi(f)$ L^2-varying. By Lemma 3.3, $\mathbb{E}[u|\pi(f_l)] \to \mathbb{E}[u|\pi(f)]$ strongly as $l \to \infty$ for any $u \in L^2(\mathcal{F})$. It follows that

$$\lim_l (\mathcal{I}(f_l)(u))_1 = \lim_l \langle f_l, \mathbb{E}[u|\pi(f_l)]\rangle = \langle f, \mathbb{E}[u|\pi(f)]\rangle = (\mathcal{I}(f)(u))_1. \tag{4.2}$$

Also, by L^2-varying convergence

$$\lim_l (\mathcal{I}(f_l)(u))_2 = \lim_l \|\mathbb{E}[u|\pi(f_l)]\|_2 = \|\mathbb{E}[u|\pi(f)]\|_2 = (\mathcal{I}(f)(u))_2. \tag{4.3}$$

Combining (4.2) and (4.3), yields the desired continuity. $\square$

Lemma 4.6. *The map $\mathcal{I}^{-1} : \mathcal{K} \to \mathbb{H}_1$ is continuous, where $\mathcal{K} := \mathcal{I}(\mathbb{H}_1)$ carries the relative topology inherited from $\mathbb{T}$ and $\mathbb{H}_1$ is equipped with the weak topology.*

Proof. We note that L^2-varying, strong and weak convergence respectively are well-defined for nets and the topology generated by nets coincides with the one generated by sequences, see Lemma A.3 (vi) in the appendix for details. Let $\{x_i\}_{i\in I}$ be a convergent net of elements in $\mathcal{K} \subset \mathbb{T}$ such that its limit satisfies $x := \lim_{i\in I} x_i \in \mathcal{K}$. Set $f := \mathcal{I}^{-1}(x)$ as well as $f_i := \mathcal{I}^{-1}(x_i)$, $i \in I$. Let $L^2(\mathcal{F})$. Then,

$$\begin{aligned}
&\lim_{i\in I} \left(\langle f_i, \mathbb{E}[u|\pi(f_i)]\rangle, \|\mathbb{E}[u|\pi(f_i)]\|_2\right) \\
=&\lim_{i\in I} \mathcal{I}(f_i)(u) = \lim_{i\in I} x_i(u) = x(u) = \mathcal{I}(f)(u) \\
=&\left(\langle f, \mathbb{E}[u|\pi(f)]\rangle, \|\mathbb{E}[u|\pi(f)]\|_2\right).
\end{aligned} \tag{4.4}$$

Clearly, as u was arbitrary, $\pi(f_i) \to \pi(f)$ in the L^2-varying sense. However, $\|f_i\|_2 \leq 1$, $i \in I$ and $\|f\|_2 \leq 1$.

Let us verify the weak convergence. Denote $v := \mathbb{E}[u|\pi(f)]$. Let $v_i \in L^2(\pi(f_i))$, $i \in I$, $v \in L^2(\pi(f))$ such that $v_i \to v$ strongly in τ-topology. Clearly,

$$\begin{aligned}
|\langle f_i, v_i\rangle - \langle f, v\rangle| \leq& |\langle f, v_i\rangle - \langle f_i, \mathbb{E}[u|\pi(f_i)]\rangle| + |\langle f_i, \mathbb{E}[u|\pi(f_i)]\rangle - \langle f, v\rangle| \\
\leq& \|f_i\|_2 \|v_i - \mathbb{E}[u|\pi(f_i)]\|_2 + |\langle f_i, \mathbb{E}[u|\pi(f_i)]\rangle - \langle f, \mathbb{E}[u|\pi(f)]\rangle|,
\end{aligned}$$

where the first term converges to zero by Lemma 3.3 and Lemma 3.8 (ii) and the second term converges to zero by (4.4). However, since u was arbitrary, we obtain the result for all $v \in L^2(\pi(f))$ and thus $f_i \rightharpoonup f$ in the weak sense. $\square$

Proposition 4.7. *$\mathcal{K} := \mathcal{I}(\mathbb{H}_1)$ is closed in $\mathbb{T}$.*

Proof. We need to verify that for any net $\{x_i\}_{i\in I}$ of elements $x_i \in \mathcal{K}$, $i \in I$, we have that all its limit points are contained in $\mathcal{K}$. Let $x \in \mathbb{T}$ be some limit point of $\{x_i\}_{i\in I}$. Then a subnet $\{x_j\}_{j\in J}$ of $\{x_i\}_{i\in I}$ converges to x. Set $f_j := \mathcal{I}^{-1}(x_j)$, $j \in J$. We have that $\|f_j\|_2 \le 1$ for $j \in J$. Based on (4.1), define the functional

$$f^x(u) := (x(u))_1, \quad u \in L^2(\mathcal{F}),$$

and the form

$$a^x(u,v) := \frac{1}{4}\left[(x(u+v))_2^2 - (x(u-v))_2^2\right], \quad u,v \in L^2(\mathcal{F}), \tag{4.5}$$

which is induced by the polarization of the second component of x. Note that f^x and a^x depend on the subnet $\{x_j\}_{j\in J}$ and thus on the directed set J. Our aim is to identify a unique element $f \in \mathbb{H}_1$ and its "index σ-algebra" $\pi(f)$ such that we have weak convergence $\sigma\text{-}\lim_{j\in J} f_j = f$ and thus L^2-varying convergence $\pi(f_j) \to \pi(f)$ and such that $\mathcal{I}(f) = x$. The functional f^x is a candidate for a functional f with these properties. As limits preserve linearity, f^x is linear on $L^2(\mathcal{F})$. Also,

$$|f^x(u)| \le \left|\lim_{j\in J}\langle f_j, \mathbb{E}[u|\pi(j)]\rangle\right| \le \sup_{j\in J}\|f_j\|_2\|u\|_2 \le \|u\|.$$

Hence f^x can be identified with an element in $L^2(\mathcal{F})$ which we denote by the same symbol.

Claim 1. The map $a^x : L^2(\mathcal{F}) \times L^2(\mathcal{F}) \to \mathbb{R}$ is a symmetric and non-negative definite bilinear form that satisfies

$$|a^x(u,v)| \le \|u\|_2\|v\|_2.$$

Proof of Claim 1. In fact, in analogy to (4.5), we also define

$$a_j(u,v) := a^{x_j}(u,v) := \frac{1}{4}\left[(x_j(u+v))_2^2 - (x_j(u-v))_2^2\right], \quad j \in J.$$

Thus $x_j = \mathcal{I}(f_j)$ induces a symmetric, non-negative definite bilinear form a_j on $L^2(\mathcal{F})$ such that by polarization

$$a_j(u,v) = \langle \mathbb{E}[u|\pi(f_j)], v\rangle = \langle \mathbb{E}[u|\pi(f_j)], \mathbb{E}[v|\pi(f_j)]\rangle$$

for all $u, v \in L^2(\mathcal{F})$. The corresponding properties for the elements x_j, $j \in J$ yield the claim after passing to the limit. $\square$

By Kato [22, Chapter V.2], there exists a unique bounded linear operator T^x with domain $L^2(\mathcal{F})$ and $\|T^x\|_{L(L^2(\mathcal{F}))} \le 1$ such that $a^x(u,v) = \langle T^x u, v\rangle$ for every $u, v \in L^2(\mathcal{F})$. Furthermore, we see that T^x is non-negative definite and symmetric. Define also $T_j u := T^{x_j} u := \mathbb{E}[u|\pi(f_j)]$, $u \in L^2(\mathcal{F})$.

At this point, we need to prove that T^x is a projection, that is, $(T^x)^2 = T^x$ on $L^2(\mathcal{F})$. However, in order to avoid double limits, we shall study the range of T^x. To this end, consider the set

$$\mathcal{M}^x := \{u \in \mathcal{B}_b(\Omega, \mathcal{F}) \mid \langle T^x u, v\rangle = \langle u, v\rangle \text{ for every } v \in L^2(\mathcal{F})\},$$

where $\mathcal{B}_b(\Omega, \mathcal{F})$ is the space of all bounded, $\mathcal{F}$-measurable real-valued maps on Ω.

Claim 2. The set $\mathcal{M}^x$ is a linear subspace of $\mathcal{B}_b(\Omega, \mathcal{F})$ closed under uniform convergence and bounded monotone convergence such that $1_\Omega \in \mathcal{M}^x$.

Proof of Claim 2. Since $T_j 1_\Omega = \mathbb{E}[1_\Omega|\pi(f_j)] = 1_\Omega$ for every $j \in J$, we have for every $v \in L^2(\mathcal{F})$ that

$$\langle 1_\Omega, v\rangle = \lim_{j\in J}\langle T_j 1_\Omega, v\rangle = \langle T^x 1_\Omega, v\rangle$$

and thus $1_\Omega \in \mathcal{M}^x$. We see that $\mathcal{M}^x$ is closed under uniform convergence as follows. Let $u_n \in \mathcal{M}^x$, $n \in \mathbb{N}$ and let $u : \Omega \to \mathbb{R}$ be bounded such that $\|u_n - u\|_\infty := \sup_{\omega\in\Omega} |u_n(\omega) - u(\omega)| \to 0$. Then u is $\mathcal{F}$-measurable and for all $v \in L^2(\mathcal{F})$,

$$\begin{aligned}
|\langle u, v\rangle - \langle T^x u, v\rangle| \leq& |\langle u, v\rangle - \langle u_n, v\rangle| \\
&+ \underbrace{|\langle u, v\rangle - \langle T^x u_n, v\rangle|}_{=0 \text{ for every } n\in\mathbb{N}} \\
&+ |\langle T^x u_n, v\rangle - \langle T^x u, v\rangle| \\
\leq& \|v\|_2 (1 + \|T^x\|_{L(L^2(\mathcal{F}))})\|u_n - u\|_\infty \to 0.
\end{aligned}$$

In order to see that $\mathcal{M}^x$ is closed under bounded monotone limits, let $u_n \in \mathcal{M}^x$, $n \in \mathbb{N}$ such that $u_n \geq 0$ and $u_n \uparrow u$, where u is bounded. Clearly, u must be $\mathcal{F}$-measurable. Let $v \in L^2(\mathcal{F})$ with $v \geq 0$ $\mathbb{P}$-a.s. By a limit procedure, we see that T^x is a positivity preserving operator, that is, $T^x v \geq 0$ $\mathbb{P}$-a.s. By symmetry of T^x and the monotone convergence theorem,

$$\sup_n \langle T^x u_n, v\rangle = \langle T^x v, \sup_n u_n\rangle = \langle T^x v, u\rangle = \langle T^x u, v\rangle.$$

The case of general $v \in L^2(\mathcal{F})$ follows by splitting $v = v^+ - v^-$ into positive and negative parts respectively. □

Claim 3. The space $\mathcal{M}^x$ is an algebra with respect to pointwise multiplication.

Proof of Claim 3. Let $u, w \in \mathcal{M}^x$. Clearly, uw is bounded and $\mathcal{F}$-measurable. By symmetry, for $v \in L^\infty(\mathcal{F})$,

$$\begin{aligned}
\langle T^x(uw), v\rangle =& \langle uw, T^x v\rangle = \lim_{j\in J}\langle uw, T_j v\rangle = \lim_{j\in J}\langle T_j(uw), v\rangle \\
=& \lim_{j\in J}\langle T_j(uT_j w), v\rangle = \lim_{j\in J}\langle uT_j w, T_j v\rangle \\
=& \lim_{j\in J}\langle T_j^2 w, uT_j v\rangle = \lim_{j\in J}\langle T_j w, T_j(uT_j v)\rangle = \lim_{j\in J}\langle T_j w, T_j(uv)\rangle \\
=& \lim_{j\in J}\langle T_j w, uv\rangle = \langle T^x w, uv\rangle = \langle uw, v\rangle
\end{aligned}$$

where we have repeatedly used symmetry, idempotence of T_j and the following $\mathbb{P}$-a.s. tower-type property for bounded functions u, w:

$$T_j(uT_j w) = T_j u T_j w = T_j(uw),$$

see Schilling [35, Theorem 22.5 (iii)] and Moy [31, Property T'3]. The proof of the claim is concluded by approximating $v \in L^2(\mathcal{F})$ by elements in $L^\infty(\mathcal{F})$. □

By the monotone class theorem, cf. Bogachev [10, Theorem 2.12.9 (ii), p. 146], and by Claims 2 and 3, we have that $\mathcal{B}_b(\Omega, \sigma(\mathcal{M}^x)) \subset \mathcal{M}^x$, where $\sigma(\mathcal{M}^x) =: \Sigma$ is the σ-algebra generated by $\mathcal{M}^x$. Hence, using Schilling [35, Theorem 22.5 (iii)], we infer that the $\|\cdot\|_2$-closure of $\mathcal{M}^x$ in $L^2(\mathcal{F})$ is equal to $L^2(\Sigma)$ and the property

$$T^x u = u$$

holds for every $u \in L^2(\Sigma)$. Thus T^x is the orthogonal projection on $L^2(\Sigma)$, in particular, $T^x v = \mathbb{E}[v|\Sigma]$ for every $v \in L^2(\mathcal{F})$.

It remains to identify the limit. Firstly, as above, $T_j \xrightarrow[j\in J]{} T^x$ converges in the weak operator topology in $L^2(\mathcal{F})$. Since T^x is a projection, we have that $T_j \xrightarrow[j\in J]{} T^x$ converges in the strong operator topology of $L^2(\mathcal{F})$ e.g. by Halmos [20, Problem 115, p. 62]. Thus $\pi(f_j) \xrightarrow[j\in J]{} \Sigma$ converges in the L^2-varying sense. On the other hand, we have for every $v \in L^2(\mathcal{F})$,

$$\begin{aligned}\langle f^x, v\rangle = (x(v))_1 = \lim_{j\in J}(x_j(v))_1 \\ = \lim_{j\in J}\langle f_j, \mathbb{E}[v|\pi(f_j)]\rangle = \lim_{j\in J}\langle \mathbb{E}[f_j|\pi(f_j)], \mathbb{E}[v|\pi(f_j)]\rangle.\end{aligned}$$

In particular, if $v \in L^2(\Sigma)$, we get the desired weak convergence in σ-topology $f_j \rightharpoonup f^x$ for $j \in J$ and $\Sigma = \pi(f^x)$. Hence $\mathcal{I}(f^x) = x$.

The proof is complete. □

As a direct consequence of Theorem 4.1, we give a new criterion for the existence of a converging σ-algebra. In contrast to Doob's martingale convergence theorem, no monotonicity on the sequence of σ-algebras is required. We recall an exact notion of a limit point from Tsukada [40, p. 137], (Property (E)) for the present case $p = 2$.

(E) There exists $\mathcal{B}_0 \in \mathbb{F}^*$ such that $||E[u|\mathcal{B}_n] - E[u|\mathcal{B}_0]||_2 \to 0$ for every $u \in L^2(\mathcal{F})$.

Corollary 4.8. *Let $\mathcal{B}_k \in \mathbb{F}^*$, $k \in \mathbb{N}$. Property (E) from Tsukada [40] is implied by the following property:*

$$+\infty > \liminf_k \|\mathbb{E}[u|\mathcal{B}_k]\|_2 \geq \limsup_k \|\mathbb{E}[u|\mathcal{B}_k]\|_2 \quad \text{for every} \quad u \in L^2(\mathcal{F}). \tag{4.6}$$

Proof. By Theorem 4.1, for any subsequence $\{\mathcal{B}_{k_l}\}$ of $\{\mathcal{B}_k\}$ there exists a non-relabeled subsequence $\{\mathcal{B}_{k_l}\}$ and a sub-σ-algebra $\mathcal{B}_0 \in \mathbb{F}^*$ such that $\mathcal{B}_{k_l} \to \mathcal{B}_0$ in the L^2-varying sense as $l \to \infty$. However, by (4.6), $\lim_l \|\mathbb{E}[u|\mathcal{B}_{k_l}]\|_2$ exists for any $u \in L^2(\mathcal{F})$. Let $\mathcal{B}_1$ be any other possible limit of the subsequence $\{\mathcal{B}_{k_l}\}$. We get by L^2-varying convergence and (4.6) that

$$\|\mathbb{E}[u|\mathcal{B}_0]\|_2 = \lim_l \|\mathbb{E}[u|\mathcal{B}_{k_l}]\|_2 = \|\mathbb{E}[u|\mathcal{B}_1]\|_2.$$

Since the above identity holds for any $u \in L^2(\mathcal{F})$, we get that $\mathcal{B}_0 = \mathcal{B}_1$ up to $\mathbb{P}$-negligible subsets, compare with the proof of Proposition 2.4. Since this argument can be repeated for any subsequence, we get that the initial sequence converges $\mathcal{B}_k \to \mathcal{B}_0$ in the L^2-varying sense. □

5. Probabilistic properties of L^2-varying convergence

We show some useful implications that follow from our convergence of σ-algebras.

5.1. **Continuity of conditional probability measures in σ-algebra.** For simplicity, we stick to the conditioning of the identity map $X : \Omega \to \Omega$, $X(\omega) = \omega$ and impose some structure on Ω.

We consider the regular conditional probability $\mathbb{P} : \Omega \times \mathcal{F} \times \mathbb{F}^* \to [0, 1]$ that depends now additionally on the conditioning σ-algebra and satisfies for each $\omega \in \Omega$, $\mathcal{B} \in \mathbb{F}^*$, $A \in \mathcal{F}$:

(i) $\mathbb{P}_\omega(\cdot|\mathcal{B})$ is a probability measure on $(\Omega, \mathcal{F})$.
(ii) $\omega \mapsto \mathbb{P}_\omega(A|\mathcal{B})$ is $\mathcal{B}$-measurable.
(iii) We have $\mathbb{P}_\omega(A|\mathcal{B}) = \mathbb{E}[1_A|\mathcal{B}](\omega)$ almost surely.

If Ω is a Polish space, the conditional probability $(\omega, A) \mapsto \mathbb{P}_\omega(A|\mathcal{B})$ exists for every $\mathcal{B} \in \mathbb{F}^*$, see Faden [18] for a characterization of existence. Before we move to the continuity of conditional probabilities, we state first a simple but useful result.

Lemma 5.1. *$\mathcal{B}_n \to \mathcal{B}$ in the L^2-varying sense implies $\mathbb{E}[g|\mathcal{B}_n] \to \mathbb{E}[g|\mathcal{B}]$ in probability for all $g \in L^2(\mathcal{F})$.*

Proof. For all $g \in L^2(\mathcal{F})$ we have $\mathbb{E}[g|\mathcal{B}_n] \to \mathbb{E}[g|\mathcal{B}]$ in $L^2(\mathcal{F})$ and hence in probability. □

Remark 5.2. Let $\mathcal{B}_n, \mathcal{B} \in \mathbb{F}^*$, $n \in \mathbb{N}$ and $B \in \mathcal{B}$, then clearly $1_B \in L^2(\mathcal{F})$. By Lemma 5.1, L^2-varying convergence $\mathcal{B}_n \to \mathcal{B}$ implies the *Skorohod J_1 convergence*, that is $\mathbb{E}[1_B|\mathcal{B}_n] \to \mathbb{E}[1_B|\mathcal{B}]$ for every $B \in \mathcal{B}$, see Coquet et al. [13].

We have an almost sure weak continuity, with respect to the conditioning information, of the conditional probability.

Proposition 5.3. *Let Ω be a Polish space and set $\mathcal{B}(\Omega) = \mathcal{F}$. Let $\mathcal{B}_n \to \mathcal{B}$ in the L^2-varying sense. Suppose that for each $f \in C_b(\Omega)$, there exists a sequence $\varepsilon_n = \varepsilon_n(f) > 0$, $n \in \mathbb{N}$ with $\varepsilon_n \searrow 0$ and that*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|\mathbb{E}[f|\mathcal{B}_n] - \mathbb{E}[f|\mathcal{B}]| > \varepsilon_n\right) < \infty.$$

Then we have $\mathbb{P}_\omega(\cdot|\mathcal{B}_n) \rightharpoonup \mathbb{P}_\omega(\cdot|\mathcal{B})$ weakly with respect to $C_b(\Omega)$ for $\mathbb{P}$-almost every $\omega \in \Omega$.

Proof. Let $f \in C_b(\Omega)$. Since $C_b(\Omega) \subset L^2(\mathcal{F})$, we have by the L^2-varying convergence of $\mathcal{B}_n$ to $\mathcal{B}$, Lemma 5.1 and an application of the Borel-Cantelli lemma that $\mathbb{E}[f|\mathcal{B}_n] \to \mathbb{E}[f|\mathcal{B}]$ $\mathbb{P}$-a.s. As a consequence, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\int_\Omega f(\bar{\omega})\mathbb{P}_\omega(d\bar{\omega}|\mathcal{B}_n) = \mathbb{E}[f|\mathcal{B}_n](\omega) \to \mathbb{E}[f|\mathcal{B}](\omega) = \int_\Omega f(\bar{\omega})\mathbb{P}_\omega(d\bar{\omega}|\mathcal{B}),$$

and the result follows. □

5.2. **Stability of independence L^2-varying convergence.** We show that independence of σ-algebras is a robust property when moving to the L^2-varying limit.

Proposition 5.4. *Fix $\mathcal{A}_n, \mathcal{A}, \mathcal{B}_n, \mathcal{B} \in \mathbb{F}^*$, $n \in \mathbb{N}$, with $\mathcal{A}_n \to \mathcal{A}$ and $\mathcal{B}_n \to \mathcal{B}$ in the L^2-varying sense such that $\mathcal{A}_n$ and $\mathcal{B}_n$ are $\mathbb{P}$-independent for every $n \in \mathbb{N}$. Then $\mathcal{A}$ and $\mathcal{B}$ are also $\mathbb{P}$-independent.*

Proof. By Kudō [26, Theorem 3.2], we have

$$\underline{\mathcal{B}} := \liminf_n \mathcal{B}_n = \left\{A \in \mathcal{F} : \lim_n \inf_{B \in \mathcal{B}_n} \mathbb{E}[1_{A\Delta B}] = 0\right\},$$

where $A\Delta B$ denotes the symmetric difference of the events A and B. As shown in Alonso and Brambila-Paz [3, Lemma 1.1 and Lemma 1.2], $\mathcal{B} \subset \underline{\mathcal{B}}$ characterizes weak convergence of $\{\mathcal{B}_n\}$, that is, $\mathbb{E}[1_A|\mathcal{B}_n] \to \mathbb{E}[1_A|\mathcal{B}]$ in probability for all $A \in \mathcal{F}$, which is implied via Lemma 5.1 by the L^2-varying convergence. We may assume $\mathcal{B} = \underline{\mathcal{B}}$ and $\mathcal{A} = \underline{\mathcal{A}}$ and take $A \in \mathcal{A}$ and $B \in \mathcal{B}$ arbitrary. Then there are sets $A_n \in \mathcal{A}_n$ and $B_n \in \mathcal{B}_n$ for each n such that

$$\lim_n \mathbb{E}[1_{A_n \Delta B_n}] = 0.$$

Via the general identity $|1_C - 1_{C'}| = 1_{C\Delta C'}$ for all $C, C' \in \mathcal{F}$, we derive

$$\begin{aligned}\mathbb{E}[|1_{A\cap B} - 1_{A_n\cap B_n}|] &= \mathbb{E}[|1_{A\cap B} - 1_{A_n\cap B} + 1_{A_n\cap B} - 1_{A_n\cap B_n}|] \\ \leq &\mathbb{E}[1_{A\Delta A_n}] + \mathbb{E}[1_{B\Delta B_n}]\end{aligned}$$

and also $\mathbb{E}[|1_A 1_B - 1_{A_n} 1_{B_n}|] \leq \mathbb{E}[1_{A\Delta A_n}] + \mathbb{E}[1_{B\Delta B_n}]$, due to the independence via $\mathbb{E}[1_{A_n} 1_{B_n}] = \mathbb{E}[1_{A_n\cap B_n}]$. Consequently, we have

$$\mathbb{E}[1_{A_n}] \cdot \mathbb{E}[1_{B_n}] - \mathbb{E}[1_A 1_B] \to 0 \quad \text{and} \quad \mathbb{E}[1_{A\cap B}] - \mathbb{E}[1_A] \cdot \mathbb{E}[1_B] \to 0$$

and the result follows. □

Compare also with Vidmar [42, Section 3].

5.3. **Join and meet operations.** Finally, we show that the lattice operations on $\mathbb{F}^*$ turn out to be continuous under the L^2-varying topology.

Proposition 5.5. *The join operation $\mathcal{B} \vee \mathcal{C} = \sigma(\mathcal{B}, \mathcal{C})$ and the meet operation $\mathcal{B} \wedge \mathcal{C} = \mathcal{B} \cap \mathcal{C}$, for some $\mathcal{B}, \mathcal{C} \in \mathbb{F}^*$ are L^2-varying jointly continuous operations $\vee, \wedge : \mathbb{F}^* \times \mathbb{F}^* \to \mathbb{F}^*$.*

Proof. $\vee$: By Remark 5.2, the joint continuity of $\vee$ follows from Coquet et al. [13, Proposition 2.3].

$\wedge$: Let $\mathcal{B}_n, \mathcal{C}_n, \mathcal{B}, \mathcal{C} \in \mathbb{F}^*$ with $\mathcal{C}_n \to \mathcal{C}$ and $\mathcal{B}_n \to \mathcal{B}$ in the L^2-varying sense. Set $\mathcal{C}_\vee = \bigvee_{n\in\mathbb{N}} \mathcal{C}_n$. The associated orthogonal projections $P^{\mathcal{A}}(\cdot) = \mathbb{E}[\cdot|\mathcal{A}]$, by setting $f^C = P^{\mathcal{C}_\vee}(f)$ and $f^B = P^{\mathcal{B}}(f)$, satisfy for all $f \in L^2(\mathcal{F})$ the following

$$\begin{aligned}
& ||P^{\mathcal{B}_n} \circ P^{\mathcal{C}_n}(f) - P^{\mathcal{B}} \circ P^{\mathcal{C}}(f)||_2 \\
\leq\ & ||P^{\mathcal{B}_n} \circ P^{\mathcal{C}_n}(f) - P^{\mathcal{B}} \circ P^{\mathcal{C}_n}(f)||_2 + ||P^{\mathcal{B}} \circ P^{\mathcal{C}_n}(f) - P^{\mathcal{B}} \circ P^{\mathcal{C}}(f)||_2 \\
\leq\ & ||P^{\mathcal{C}_\vee} \circ (P^{\mathcal{B}_n} - P^{\mathcal{B}})(f)||_2 + ||P^{\mathcal{B}} \circ (P^{\mathcal{C}_n} - P^{\mathcal{C}})(f)||_2 \\
\leq\ & \underbrace{||(P^{\mathcal{B}_n} - P^{\mathcal{B}})(f^C)||_2}_{\to 0} + \underbrace{||(P^{\mathcal{C}_n} - P^{\mathcal{C}})(f^B)||_2}_{\to 0},
\end{aligned}$$

where we use the commutativity of projections. Consequently, $\mathcal{B}_n \wedge \mathcal{C}_n$ converges to $\mathcal{B} \wedge \mathcal{C}$ in the L^2-varying sense. ☐

6. Application to information economics

In economics, sub-σ-algebras often serve as a model of (incomplete) information of some decision maker (DM). Based on Section 4, we can consider problems of information design, a recent field in theoretical economics that departs from the idea of mechanism design (inverse game theory, see Bergemann and Morris [8] for a Bayesian approach). In such models there is a second better informed agent, the so called omniscient information designer (ID). The ID can transfer information to the DM. The optimal and payoff relevant decision of the DM, after receiving information from the ID, then also affects the payoff of the ID. The topology of L^2-varying convergence establishes a setting that allows to analyze problems of strategic information transfer.

6.1. **Strategies of the DM.** For the DM, let there be a finite set of actions $A = \{a_1, \dots, a_N\}$ that determines the payoff. A (pure) strategy is a mapping $\mathtt{s} : \Omega \to A$. A mixed strategy is given by $s : \Omega \to \Delta_A$, where Δ_A denotes the simplex in $\mathbb{R}^{|A|}$ and models the set of all mixed actions. The measurability condition on s now constraints the DM's set of feasible strategies. As the imperfectly informed DM fails to be omniscient, she is only endowed with a sub σ-algebra $\mathcal{G} \subsetneq \mathcal{F}$ as information. Without information transfer the set of information feasible strategies is then

$$L(\mathcal{G}) = \Big\{ s : \Omega \to \Delta_A : s \text{ is } \mathcal{G}\text{-measurable and } \mathbb{E}^{\mathbb{P}}[s] < \infty \Big\}.$$

6.2. **(Randomized) σ-algebras as information.** By Proposition 2.4 and Theorem 4.1, we have that $(\mathbb{F}^*, \kappa)$ is a metrizable compact topological space. In turn, this is equivalent to the weak* compactness of the space of all probability measures on $\mathbb{F}^*$, which we denote by

$$\Delta(\mathbb{F}^*) := \mathcal{M}_1(\mathbb{F}^*, \mathcal{B}(\mathbb{F}^*))$$

where the weak* topology is given by $\sigma(\Delta(\mathbb{F}^*), C_b(\mathbb{F}^*))$ and $C_b(\mathbb{F}^*)$ denotes the space of continuous bounded real valued functions on $\mathbb{F}^*$. For details and the stated equivalence, see Aliprantis and Border [1, Chapter 15].

The ID knows $\mathcal{F}$ and can send parts of his information to the DM. We also allow for a probabilistic transfer of information. Some $\nu \in \Delta(\mathbb{F}^*)$ is then interpreted as a randomized information transfer, that is, the probability that one of the σ-algebras $\mathcal{G} \in \mathbb{G} \in \mathcal{B}(\mathbb{F}^*)$ is received by the DM and is exactly captured by ν. For instance, the class of most elementary information transfers, consists of Dirac measures $\delta_{\mathcal{G}} \in \Delta(\mathbb{F}^*)$, for some $\mathcal{G} \in \mathbb{F}^*$, and means a deterministic information transfer of $\mathcal{G}$.

6.3. **Information design.** For simplicity, let the only action of the ID be to transfer information. The expected payoff of the ID then depends on the optimal strategy s^* of the DM, which in turn depends on the received information ν. Under some increasing, continuous and concave Bernoulli utility index $v : \mathbb{R} \to \mathbb{R}$ the ID's optimization problem is given by

$$\max_{\nu \in \Delta(\mathbb{F}^*)} \int_{\mathbb{F}^*} \mathbb{E}^{\mathbb{P}}[v(s^*(\cdot, \mathcal{H})] \, d\nu(\mathcal{H}),$$

where $s^*(\cdot, \mathcal{H}) : \Omega \to \Delta_A$ is some $\mathcal{G} \vee \mathcal{H}$-measurable and integrable payoff relevant strategy that is chosen by the uninformed DM. As such, the maximization problem is not yet well-posed.

The reason for this stems from the strategy s^* of the DM which depends in turn on the information transfer ν as it is solution of optimization with constraints depending on ν. We clarify this in the next subsection.

6.4. **Uninformed decision maker.** Suppose the ID sends the randomized information transfer ν to the DM. This allows the DM to consider a larger space of informationally feasible strategies $L^2(\mathcal{G}; \nu)$, that is, the mixture of all possible realized information transfers. For some realization $\mathcal{H}$, the DM is now equipped with $\mathcal{G} \vee \mathcal{H}$ and an enlarged set of feasible strategies $L(\mathcal{G} \vee \mathcal{H})$.

Ex ante, the DM incorporates now all reactions to possible information transfers:

$$L(\mathcal{G}; \mu) = \Big\{ s : \Omega \times \mathbb{F}^* \to \Delta_A : s(\cdot, \mathcal{H}) \in L(\mathcal{G} \vee \mathcal{H}), \mathbb{E}^{\mathbb{P} \otimes \mu}[s^2]^{1/2} < \infty \Big\},$$

where $\mu \in \Delta(\mathbb{F}^*)$ denotes his a priori given belief about the likelihood of information transfers.

With this and for a $\nu \in \Delta(\mathbb{F}^*)$ chosen by the ID, the DM's optimization problem (sharing the utility index with the ID) is given by

$$\max_{s \in L(\mathcal{G}; \mu)} \int_{\mathbb{F}^*} \mathbb{E}^{\mathbb{P}} \Big[v\big(s(\cdot, \mathcal{H})\big) \Big] \, d\nu(\mathcal{H}).$$

6.5. **Solution of the information design problem.** Having specified the perspective of the DM and ID, we introduce an equilibrium concept, when both players are interacting in a strategic way. Therefore, we set $f(s, \nu) = \int_{\mathbb{F}^*} \mathbb{E}^{\mathbb{P}}[v(s(\cdot, \mathcal{H}))] d\nu(\mathcal{H})$.

Definition 6.1. A game theoretic *equilibrium of the information design problem* is a pair $(\hat{\nu}, \hat{s}) \in \Delta(\mathbb{F}^*) \times L(\mathcal{G}; \mu)$ such that $f(s, \hat{\nu}) \leq f(\hat{s}, \hat{\nu})$ for all $s \in L(\mathcal{G}; \mu)$ and $f(\hat{s}, \nu) \leq f(\hat{s}, \hat{\nu})$ for all $\nu \in \Delta(\mathbb{F}^*)$.

Based on Theorem 4.1, we have existence of an equilibrium.

Proposition 6.2. *A game theoretic equilibrium of the information design problem exists.*

Proof. Let us set $\sigma_{\mathcal{F}} := \sigma(L^2(\mathcal{F}; \mathbb{R}^{|A|}), L^2(\mathcal{F}; \mathbb{R}^{|A|})^*)$ and $\sigma_{\mathcal{F} \otimes \mathcal{B}(\mathbb{F}^*)} := \sigma(L^2(\mathcal{F} \otimes \mathcal{B}(\mathbb{F}^*); \mathbb{R}^{|A|}), L^2(\mathcal{F} \otimes \mathcal{B}(\mathbb{F}^*); \mathbb{R}^{|A|})^*)$. For each $\mathcal{G} \in \mathbb{F}^*$, the set $L(\mathcal{G})$ is bounded, closed and convex and hence $\sigma_{\mathcal{F}}$-compact in the space of $\mathbb{R}^{|A|}$-valued random variables $L^2(\mathcal{F}; \mathbb{R}^{|A|})$. The space $L^2(\mathcal{F} \otimes \mathcal{B}(\mathbb{F}^*); \mathbb{R}^{|A|})$, with product measure $\mathbb{P} \otimes \mu$, has the same property, and its subset $L(\mathcal{G}; \mu)$ is a $\sigma_{\mathcal{F} \otimes \mathcal{B}(\mathbb{F}^*)}$-closed, convex and bounded set. By Berge's maximum theorem, Aliprantis and Border [1, Theorem 17.31.], the correspondence

$$S(\nu) = \arg\max_{s \in L(\mathcal{G}; \mu)} f(s, \nu)$$

is upper-hemicontinuous. Moreover, for any ν, $S(\nu)$ is non-empty, convex and $\sigma_{\mathcal{F} \otimes \mathcal{B}(\mathbb{F})}$-compact, by the concavity and continuity of $f(\cdot, \nu)$.

On the ID side and again by Berge's maximum theorem the correspondence

$$V(s) = \arg\max_{\nu \in \Delta(\mathbb{F}^*)} f(s, \nu).$$

is upper-hemicontinuous, non-empty, convex and $\sigma(\Delta(\mathbb{F}^*), C_b(\mathbb{F}^*))$-compact valued. The upper hemicontinuity with compact values of the product of the above correspondences, given by $SV : \Delta(\mathbb{F}^*) \times L(\mathcal{G}; \mu) \Rightarrow \Delta(\mathbb{F}^*) \times L(\mathcal{G}; \mu)$ follows from Aliprantis and Border [1, Theorem 17.28.1.]. By Aliprantis and Border [1, Theorem 17.10.2.], SV has a closed graph, as the space the space of of signed measures $\mathcal{M}(\mathbb{F}^*, \mathcal{B}(\mathbb{F}^*))$ equipped with the topology of weak convergence is a locally convex topological vector space.

An application of the Kakutani-Fan-Glicksberg fixed-point theorem, see Aliprantis and Border [1, Corollary 17.55] gives a $(\hat{s}, \hat{\nu}) \in SV(\hat{s}, \hat{\nu})$, that is, the desired equilibrium exists. □

Appendix A. Sequential spaces, convergence and proofs

Let $(T, \mathcal{T})$ be a topological space. We assume that the reader is familiar with the notions net, subnet, directed set, "cofinal", "frequently", "eventually". They are e.g. explained in Engelking [17], Kelley [23].

Recall that a set A included in T is open if and only if every net $\{x_i\}_{i \in I}$ which converges to a point $x \in A$ is eventually in A. Also, a set A included in T is closed if and only if it contains with any net all its possible limits, or equivalently, no net included in A converges to a point in $T \setminus A$. For a set A included in T one defines the relative topology of A in T by $\mathcal{T}_A := \{O \cap A \mid O \in \mathcal{T}\}$. $B \subset A$ is called relatively open if $B \in \mathcal{T}_A$ and $B \subset A$ is called relatively closed if $A \setminus B \in \mathcal{T}_A$.

Recall the following basic definition.

Definition A.1. A (Kuratowski) closure operator on a set S is a mapping $\overline{} : 2^S \to 2^S$ such that the Kuratowski closure axioms

(K1) $\overline{\varnothing} = \varnothing$,
(K2) for each $A \in 2^S$: $A \subset \overline{A}$,
(K3) for each $A, B \in 2^S$: $\overline{A \cup B} = \overline{A} \cup \overline{B}$,
(K4) for each $A \in 2^S$: $\overline{(\overline{A})} = \overline{A}$,

hold.

If X is a topological space, we define for any subset $A \subset X$ the *closure* w.r.t. the topology of X as $\overline{A} := \bigcap_{\substack{B \supset A,\\ B \text{ closed}}} B$. It satisfies the Kuratowski closure axioms. Conversely, a Kuratowski closure operator on a set S defines a topology on S by saying $A \subset S$ is closed if $\overline{A} = A$. Then the Kuratowski closure operator coincides with the closure w.r.t. to the topology it generates. See e.g. Kelley [23, Chapter 1].

Definition A.2. A topological space X is called sequential space if a set $A \subset X$ is closed if and only if together with any sequence it contains all its limits. A topological space X is called a Fréchet-Urysohn space if for every $A \subset X$ and every $x \in \overline{A}$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of points of A converging to x.

We assume that the reader is familiar with the terms *first countable*, *second countable*, *compact*, *sequentially compact* and *countably compact*.

Lemma A.3. (i) *Every first-countable space is a Fréchet-Urysohn space and every Fréchet-Urysohn space is a sequential space.*
(ii) *Any subspace of a Fréchet-Urysohn space is itself a Fréchet-Urysohn space.*
(iii) *Any closed subspace of sequential space is itself a sequential space.*

(iv) *A mapping F of a sequential space X to a topological space Y is continuous if and only if $F(\lim_{n\to\infty} x_n) \subset \lim_{n\to\infty} F(x_n)$ for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in the space X.*

(v) *Sequential compactness and countable compactness are equivalent in the class of sequential spaces.*

(vi) *In a sequential space the characterization of open and closed sets found as in the beginning of this appendix holds with nets replaced by sequences.*

Proof. (i): Engelking [17, Theorem 1.6.14], (ii),(iii): Engelking [17, Exercise 2.1.H], (iv): Engelking [17, Proposition 1.6.15], (v): Engelking [17, Theorem 3.10.31], (vi): clear from the definition. □

Definition A.4. Let S be a set. A $\mathcal{L}^*$-sequential convergence or $\mathcal{L}^*$-(sequential) limit operator $\mathcal{C}$ on S is a relation between sequences $\{s_n\}_{n\in\mathbb{N}}$ of members of S and members s of S, denoted $s_n \xrightarrow[n\to\infty]{\mathcal{C}} s$ (in words: $\{s_n\}$ $\mathcal{C}$-converges to s), such that:

(L1) If $s_n = s$ for each $n \in \mathbb{N}$, then $s_n \xrightarrow[n\to\infty]{\mathcal{C}} s$.

(L2) If $s_n \xrightarrow[n\to\infty]{\mathcal{C}} s$, then $s_{n_k} \xrightarrow[k\to\infty]{\mathcal{C}} s$ for every subsequence $\{s_{n_k}\}$ of $\{s_n\}$.

(L3) If $s_n \not\xrightarrow[n\to\infty]{\mathcal{C}} s$, then $\{s_n\}$ contains a subsequence $\{s_{n_k}\}$ such that no subsequence of $\{s_{n_k}\}$ converges to s.

$\mathcal{C}$ is called a $\mathcal{S}^*$-sequential convergence or $\mathcal{S}^*$-(sequential) limit operator if additionally:

(L4) If $s_n \xrightarrow[n\to\infty]{\mathcal{C}} s$ and $s^n_m \xrightarrow[m\to\infty]{\mathcal{C}} s_n$ for each $n \in \mathbb{N}$, then there exist increasing sequences of positive integers $n_1, n_2, \ldots$ and $m_1, m_2, \ldots$ such that $s^{n_k}_{m_k} \xrightarrow[k\to\infty]{\mathcal{C}} s$.

The pair $(S, \mathcal{C})$ is called $\mathcal{L}^*$-space ($\mathcal{S}^*$-space respectively).

For a subset $A \subset S$ of a $\mathcal{L}^*$-space we define the $\mathcal{C}$-closure $\overline{A}^{\mathcal{C}} \subset S$ by the convention $s \in \overline{A}^{\mathcal{C}}$ if and only if there is a sequence $\{s_n\}$ included in A $\mathcal{C}$-converging to s.

Theorem A.5. *The $\mathcal{C}$-closure of an $\mathcal{L}^*$-sequential convergence $\mathcal{C}$ on a $\mathcal{L}^*$-space S fulfills the first three of the Kuratowski closure axioms ((K1)–(K3)). (K4) holds in addition if $\mathcal{C}$ is a $\mathcal{S}^*$-sequential convergence. In an $\mathcal{S}^*$-space S with convergence $\mathcal{C}$ the topology τ generated by the $\mathcal{C}$-closure is T_1. We have that τ-$\lim_{n\to\infty} s_n = s$ if and only if $s_n \xrightarrow[n\to\infty]{\mathcal{C}} s$, that is, convergence* a posteriori *is equivalent to the convergence* a priori.

A topology coming from an $\mathcal{L}^*$-convergence in the above sense is a sequential topology in the sense of Definition A.2. A topology coming from an $\mathcal{S}^*$-convergence in the above sense is a Fréchet-Urysohn-topology in the sense of Definition A.2. Conversely, the usual convergence of sequences in a (topological) sequential space is an $\mathcal{L}^*$-convergence and the usual convergence of sequences in a (topological) Fréchet-Urysohn space is an $\mathcal{S}^*$-convergence.

Proof. Cf. Engelking [17, Problems 1.7.18–1.7.20] and the references therein for the proof. □

Alternatively, if we impose the convention that a set $A \subset S$ is closed if and only if it contains all convergent sequences together with all their limits, this defines a T_1-topology with the property that convergence a priori is identical to convergence a posteriori even in the cases of an $\mathcal{L}^*$-space. If S is an $\mathcal{S}^*$-space this topology coincides with the one coming from the closure defined above.

A.1. **A lemma for diagonal sequences.**

Lemma A.6. *Let $\{a_{n,m}\}_{n,m\in\mathbb{N}} \subset \mathbb{R}\cup\{+\infty\}\cup\{-\infty\}$ be a doubly indexed sequence of extended real numbers. Then there exists a map $n \mapsto m(n)$ with $m(n) \uparrow +\infty$ as $n \to +\infty$ such that*

$$\liminf_n a_{n,m(n)} \geq \liminf_m \left[\liminf_n a_{n,m}\right], \tag{A.1}$$

or, equivalently

$$\limsup_n a_{n,m(n)} \leq \limsup_m \left[\limsup_n a_{n,m}\right]. \tag{A.2}$$

Proof. See Attouch and Wets [7, Appendix] or Attouch [6, Lemma 1.15 et seq.]. □

A.2. **Remaining proofs from Section 3.**

Proof of Proposition 3.6. Clearly, by Lemma 3.5, for $u_k, u \in \mathbb{H}$, $k \in \mathbb{N}$, we have that $u_k \to u$ strongly if and only if $\pi(u_k) \to \pi(u)$ in the L^2-varying sense and $u_k \to u$ in $L^2(\mathcal{F})$. Let us check (L1)–(L4) from Definition A.4.

(L1): Obvious.

(L2): This follows from the fact that $\mathbb{F}^*$ (by Proposition 2.4) and $L^2(\mathcal{F})$ (being a metric space) have property (L2).

(L3): Suppose that $u_k \not\to u$ strongly. Then $\pi(u_k) \not\to \pi(u)$ in the L^2-varying sense or $u_k \not\to u$ in $L^2(\mathcal{F})$. Suppose that $\pi(u_k) \not\to \pi(u)$ in the L^2-varying sense. Then there exists $f \in L^2(\mathcal{F})$ such that

$$\alpha_k := \|\mathbb{E}[f|\pi(u_k)]\|_2 \not\to \|\mathbb{E}[f|\pi(u)]\|_2 =: \alpha.$$

Hence, there exists a non-relabeled subsequence of $\{\alpha_k\}$, such that no subsequence of it converges to α. Also, no subsequence of $\{\pi(u_k)\}$ converges to $\pi(u)$. In the second case, there exists a non-relabeled subsequence of $\{u_k\}$ that admits no subsequence which converges to u in $L^2(\mathcal{F})$. Assume that $\pi(u_k) \to \pi(u)$. Then by Lemma 3.5, $u_k \to u$ cannot be true (for any further subsequence). However, if $\pi(u_k) \not\to \pi(u)$ we are in the first case again.

(L4): Let $u^k_m, u_k, u \in \mathbb{H}$, $k, m \in \mathbb{N}$ and suppose that $u_k \to u$ strongly as $k \to \infty$, and that for each $k \in \mathbb{N}$, let $u^k_m \to u_k$ strongly as $m \to \infty$. By an application of Lemma A.6 from the appendix, we get that there exists an increasing sequence $\{k_m\}$ of natural numbers such that

$$\begin{aligned}
&\limsup_m \left(\|u^{k_m}_m - u\|_2 + d_\kappa(\pi(u^{k_m}_m), \pi(u))\right)\\
\leq &\limsup_k \limsup_m \left(\|u^k_m - u\|_2 + d_\kappa(\pi(u^k_m), \pi(u))\right)\\
\leq &\limsup_k \limsup_m \|u^k_m - u_k\|_2 + \limsup_k \|u_k - u\|_2\\
&+ \limsup_k \limsup_m d_\kappa(\pi(u^k_m), \pi(u_k)) + \limsup_k d_\kappa(\pi(u_k), \pi(u))\\
=&0,
\end{aligned}$$

where d_κ is defined as in Proposition 2.4. We get that $u^{k_m}_m \to u$ strongly by Lemma 3.5. □

See also Tölle [38, Subchapter 5.14] for a proof in a related setup.

Proof of Proposition 3.12. Clearly, by Lemma 3.5, for $u_k \in \mathbb{H}$, $k \in \mathbb{N}$, $u \in \mathbb{H}$, we have that $u_k \rightharpoonup u$ weakly if and only if $\pi(u_k) \to \pi(u)$ in the L^2-varying sense and (W1) and (W2) from Definition 3.9 hold. Let us check (L1)–(L3) from Definition A.4.

(L1): See Remark 3.10.

(L2): Follows easily from the fact that strong convergence satisfies property (L2), see proof of Proposition 3.6.

(L3): Suppose that $u_k, u \in \mathbb{H}$, $k \in \mathbb{N}$ such that $u_k \not\rightharpoonup u$ weakly. Then $\pi(u_k) \not\rightharpoonup \pi(u)$ in the L^2-varying sense or (W1) or (W2) does not hold. Suppose that $\pi(u_k) \not\rightharpoonup \pi(u)$ in the L^2-varying sense. Then we find a non-relabeled subsequence of $\{u_k\}$ that admits no subsequence which converges weakly to u, see proof of Proposition 3.6. Assume that $\pi(u_k) \to \pi(u)$. Then (W1) or (W2) does not hold. Let us suppose that (W1) does not hold for $\{u_k\}$. Then there exists a non-relabeled subsequence such that $\lim_k \|u_k\|_2 = +\infty$, so that no subsequence of which satisfies (W1) and thus cannot converge weakly. Finally, after extracting a common subsequence for the above cases, if necessary, we assume that (W2) is violated. Then there exist $v_k, v \in \mathbb{H}$, $k \in \mathbb{N}$, with $\pi(v_k) = \pi(u_k)$ for every $k \in \mathbb{N}$ and $\pi(v) = \pi(u)$ such that $v_k \to v$ strongly and such that

$$\limsup_k |\langle u_k, v_k\rangle - \langle u, v\rangle| > 0.$$

Hence there exists a non-relabeled subsequence, such that

$$\limsup_k |\langle u_k, v_k\rangle - \langle u, v\rangle| = \lim_k |\langle u_k, v_k\rangle - \langle u, v\rangle| \in (0, +\infty],$$

such that no subsequence of it converges.

□

See also Tölle [38, Subchapter 5.6] for a proof in a related setup.

Appendix B. Proof of Claim 2 from Example 2.2

In the following we give the postponed proof of the second claim of Example 2.2. We also refer to Piccinini [33, Section 3] for a collection of similar examples.

Proof. Let $g_0(\omega) := 2\omega$, $\omega \in [0,1]$. Then, clearly, $\mathbb{E}[g_0|\mathcal{B}_0] = \mathbb{E}[g_0]1_\Omega = 2\left(\int_0^1 \omega\, d\omega\right) \cdot 1_\Omega = 1_\Omega$. We have that,

$$\mathbb{E}[g_0|\mathcal{B}_n] = \frac{\mathbb{E}[g_0 1_{I^{(n)}}]1_{I^{(n)}}}{\mathbb{P}(I^{(n)})} + \frac{\mathbb{E}[g_0 1_{(I^{(n)})^c}]1_{(I^{(n)})^c}}{(1 - \mathbb{P}(I^{(n)}))} =: G_n + H_n.$$

Firstly, denoting $m(n) := \lfloor \log_2(n) \rfloor$, $n \in \mathbb{N}$,

$$G_n = \frac{2\int_{I^{(n)}} \omega\, d\omega}{2^{-m(n)}} 1_{I^{(n)}} = \frac{2(2^{m(n)} - n)}{2^{-m(n)}} 1_{I^{(n)}},$$

secondly,

$$H_n = \frac{1 - 2\int_{I^{(n)}} \omega\, d\omega}{1 - 2^{-m(n)}} 1_{(I^{(n)})^c} = \frac{1 - 2(2^{m(n)} - n)}{1 - 2^{-m(n)}} 1_{(I^{(n)})^c}.$$

Altogether,

$$\begin{aligned}
&G_n + H_n - 1_\Omega \\
=&\frac{2(2^{m(n)} - n) - 2^{-m(n)}}{2^{-m(n)}} 1_{I^{(n)}} + \frac{1 - 2(2^{m(n)} - n) - (1 - 2^{-m(n)})}{1 - 2^{-m(n)}} 1_{(I^{(n)})^c} \\
=&\frac{2^{m(n)+1} - 2n - 2^{-m(n)}}{2^{-m(n)}} 1_{I^{(n)}} + \frac{2^{m(n)+1} - 2n + 2^{-m(n)}}{1 - 2^{-m(n)}} 1_{(I^{(n)})^c}.
\end{aligned}$$

Consider the subsequence $n_k := 2^{\lfloor \log_2 k \rfloor}$. Then, $m(n_k) = m(k)$ and

$$|G_{n_k} + H_{n_k} - 1_\Omega| = \Big| \frac{2^{m(k)+1} - 2^{m(k)+1} - 2^{-m(k)}}{2^{-m(k)}} 1_{I^{(n_k)}} + \frac{2^{m(k)+1} - 2^{m(k)+1} + 2^{-m(k)}}{1 - 2^{-m(k)}} 1_{(I^{(n_k)})^c} \Big|,$$

and thus

$$\lim_k |G_{n_k} + H_{n_k} - 1_\Omega| \leq \lim_k 1_{I^{(n_k)}} + \lim_k \left| \frac{2^{-m(k)}}{1 - 2^{-m(k)}} \right| = \lim_k 1_{I(0,m(k))} = 0, \quad \mathbb{P}\text{-a.s.}$$

Now, consider the subsequence $n_k := 2^{\lfloor \log_2 k \rfloor} + 1$. Then, $m(k) \leq m(n_k) \leq m(k) + 1$, and

$$\begin{aligned}
&G_{n_k} + H_{n_k} - 1_\Omega \\
=&\frac{2^{m(n_k)+1} - 2^{m(k)+1} - 2 - 2^{-m(n_k)}}{2^{-m(n_k)}} 1_{I^{(n_k)}} \\
&+ \frac{2^{m(n_k)+1} - 2^{m(k)+1} - 2 + 2^{-m(n_k)}}{1 - 2^{-m(n_k)}} 1_{(I^{(n_k)})^c} \\
=&(2^{2m(n_k)+1} - 2^{2m(k)+1} - 2^{1+m(n_k)} - 1) 1_{I^{(n_k)}} \\
&+ \left(\frac{2^{m(n_k)+1} - 2^{m(k)+1} - 1}{1 - 2^{-m(n_k)}} - 1 \right) 1_{(I^{(n_k)})^c},
\end{aligned}$$

and thus,

$$\begin{aligned}
&\liminf_k (G_{n_k} + H_{n_k}) \\
\leq&\limsup_k (2^{2m(n_k)+1} - 2^{2m(k)+1}) 1_{I^{(n_k)}} \\
&+ \limsup_k \left(\frac{2^{m(n_k)+1} - 2^{m(n_k)+1} - 1}{1 - 2^{-m(n_k)}} \right) 1_{(I^{(n_k)})^c} \\
\leq&- \limsup_k 1_{(I^{(n_k)})^c} \\
\leq&0,
\end{aligned}$$

and hence $\liminf_k (G_{n_k} + H_{n_k}) \neq 1_\Omega$ and thus $\mathbb{E}[g_0|\mathcal{B}_n] \not\to \mathbb{E}[g_0|\mathcal{B}_0]$ $\mathbb{P}$-a.s. □

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